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Uniqueness Theorems for Harmonic Functions Which Vanish at Lattice Points

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1. INTRODUCTION

The following theorem of Carlson [4] (or see, e.g., Boas [1, p. 153]) is well known. The complex plane is denoted by **C**.

THEOREM A. *Let f be an entire function and let* $0 < \alpha < \pi$. *If*

$$f(z) = O(e^{\alpha|z|}) \tag{1}$$

uniformly as $|z| \to \infty$ *and*

$$f(z) = 0 \qquad (z = 0, 1, 2,...),$$

then $f \equiv 0$.

Using a similar result, Boas [2] proved the following.

THEOREM B. *Let h be a real-valued harmonic function in* **C** *and let* $0 < \alpha < \pi$. *If*

$$h(z) = O(e^{\alpha|z|})$$

uniformly as $|z| \to \infty$ *and*

$$h(z) = 0 \qquad (z = 0, \pm 1, \pm 2,..., i, i \pm 1, i \pm 2,...), \tag{2}$$

then $h \equiv 0$.

Similarly, Ching [6] showed that the same conclusion holds if, in Theorem B, we replace (2) by the conditions

$$h(z) = 0 \qquad (z = 0, \pm 1, \pm 2,..., \pm i, \pm 2i,...)$$

and $h(z) = -h(-z)$ for all complex z.

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In this note, answering a question of Boas [2], we show how theorems of this type can be proved for harmonic functions in the Euclidean space $\mathbf{R}^n$ of dimension $n \geqslant 2$. I am grateful to the referee for drawing my attention to the papers of Rao [9] and Zeilberger [10] in which some of the results of the present paper are proved. The proofs given here are different from those of Rao and Zeilberger, and our methods can be applied to solve other problems.

Before stating our results, we give some notations. An arbitrary point of $\mathbf{R}^n$ is represented by $X = (x_1, ..., x_n)$, and we put

$$|X| = (x_1^2 + \cdots + x_n^2)^{1/2}$$

Throughout this note m is an integer such that $1 \leqslant m \leqslant n$, and we put

$$E^m = \{X \in \mathbf{R}^n : x_{m+1} = \cdots = x_n = 0\}$$

and

$$L^m = \{X \in E^m : x_1, ..., x_m \in \mathbf{N}\},$$

where $\mathbf{N} = \{0, 1, 2, ...\}$. Thus L^m is a copy of $\mathbf{N}^m$ embedded in E^m, and E^m is a copy of $\mathbf{R}^m$ embedded in $\mathbf{R}^n$. If G_1 and G_2 are subsets of $\mathbf{R}^n$, we put

$$G_1 + G_2 = \{X + Y : X \in G_1, Y \in G_2\}.$$

By analogy with the terminology for entire functions and harmonic functions in $\mathbf{C}$, we say that a harmonic function h in $\mathbf{R}^n$ is of exponential type α, where $0 \leqslant \alpha < \infty$ if

$$\limsup_{r \to \infty} r^{-1} \log \mathscr{M}(h; r) = \alpha,$$

where

$$\mathscr{M}(h; r) = \sup_{|X|=r} |h(X)| \qquad (r > 0).$$

Conventionally, constant functions are of exponential type 0.

Our main results are corollaries of the following theorem.

THEOREM 1. *Let h be a harmonic function of exponential type less than π in $\mathbf{R}^n$. If $h = 0$ on L^m, then $h = 0$ on E^m.*

The result fails (for any m and n) if h is of exponential type π.

The case $m = 1$ of this theorem, from which the other cases are easily deduced, is Rao's Theorem 1.3 in [9].

We come now to the applications of Theorem 1.

THEOREM 2. *Let h be a harmonic function of exponential type less than π in $\mathbf{R}^n$, and let Z_1 be a point of $\mathbf{R}^n$ whose n-th coordinate is* 1. *If $h = 0$ on $L^{n-1} \cup (L^{n-1} + \{Z_1\})$, then $h \equiv 0$.*

This is similar to Rao's Corollary 1.8 in [9] and Zeilberger's Theorem A in [10]. The case $n = 2$ of this theorem is an improvement of Theorem B quoted above.

THEOREM 3. *Let h be a harmonic function of exponential type less than π in $\mathbf{R}^n$. If $h = 0$ on the union of the n intersecting copies of $\mathbf{N}^{n-1}$ given by*

$$\{X \in L^n \colon x_i = 0 \text{ for at least one } i \ (1 \leqslant i \leqslant n)\}$$

and

$$h(X) = (-1)^{n+1} h(-X) \qquad (X \in \mathbf{R}^n), \tag{3}$$

then $h \equiv 0$.

This generalizes Ching's result, cited above.

THEOREM 4. *Let h be a harmonic function of exponential type less than π in $\mathbf{R}^n$, and let Z_0 be a fixed point of E^{n-1}. If $h = 0$ on L^{n-1} and $\partial h/\partial x_n = 0$ on $L^{n-1} + \{Z_0\}$, then $h \equiv 0$.*

This improves Zeilberger's Theorem B in [10].

In [2] Boas also asked whether a certain theorem of Cartwright on entire functions had an analogue for harmonic functions in $\mathbf{C}$. Cartwright's theorem [5] (or see [1, p. 180]) may be stated in the following form.

THEOREM C. *If f is an entire function which satisfies* (1) *for some α such that $0 < \alpha < \pi$ and there is a number M such that*

$$|f(z)| \leqslant M \qquad (z = 0, \pm 1, \pm 2, \ldots),$$

then

$$|f(z)| \leqslant \lambda(\alpha) M$$

for each z on the real axis, where λ depends only on α and is continuous on $(0, \pi)$.

By using this result and modifying the proof of Theorem 1, we shall be able to prove an analogous result for harmonic functions in $\mathbf{R}^n$. We put

$$I^m = \{X \in E^m \colon x_1, \ldots, x_m \in \mathbf{Z}\},$$

where $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

THEOREM 5. *Let h be a harmonic function of exponential type* $\alpha < \pi$ *in* $\mathbf{R}^n$. *If there is a number M such that*

$$|h(X)| \leqslant M \qquad (X \in I^m)$$

then

$$|h(X)| \leqslant \{\lambda(\alpha)\}^m M \qquad (X \in E^m).$$

The result fails (for any m and n) if h is of exponential type π.

Applying Picard's theorem that a bounded harmonic function in $\mathbf{R}^n$ is constant, we obtain immediately the following.

COROLLARY. *Let h be a harmonic function of exponential type less than* π *in* $\mathbf{R}^n$. *If h is bounded on* I^n *then h is constant.*

2. DERIVATIVES OF HARMONIC FUNCTIONS

It is well known that if h is harmonic in $\mathbf{R}^n$, then its multiple Taylor series about any point converges to h in the whole of $\mathbf{R}^n$ (see e.g. Brelot [3, p. 179]). This fact and the following result are central to the proofs of Theorems 1 and 5. We denote the origin of $\mathbf{R}^n$ by O and write

$$D^j = \partial^j/\partial x_1{}^j \qquad (j = 1, 2,...).$$

THEOREM 6. *If h is harmonic in* $\mathbf{R}^n$ *and*

$$\mathscr{M}(h; r) = O(e^{\alpha r}) \qquad (r \to \infty), \tag{4}$$

where $\alpha > 0$, *then*

$$D^j h(O) = O(j^{n-3/2}\alpha^j) \qquad (j \to \infty).$$

Before proving this result, we give some further notations. For each positive number r, we put

$$B(r) = \{X \in \mathbf{R}^n: |X| < r\}, \qquad S(r) = \{X \in \mathbf{R}^n: |X| = r\}.$$

The surface area measure on $S(r)$ is denoted by σ and the surface area of $S(1)$ by s_n.

The Poisson kernel of $B(r)$ is the function K: $B(r) \times S(r) \to \mathbf{R}$, defined by the equation

$$K(X, Y) = (s_n r)^{-1}(r^2 - |X|^2)\,|X - Y|^{-n},$$

and if h is harmonic in $\mathbf{R}^n$, we have

$$h(X) = \int_{S(r)} K(X, Y)\, h(Y)\, d\sigma(Y) \qquad (X \in B(r))$$

(see e.g. Helms [7, p. 16]). Since K and all its partial derivatives (with respect to its first argument) are continuous on $B(r) \times S(r)$, we have

$$D^j h(O) = \int_{S(r)} D^j K(O, Y)\, h(Y)\, d\sigma(Y) \qquad (j = 1, 2, \ldots). \tag{5}$$

In view of (5), we see that the main problem in the proof of Theorem 6 is to estimate $D^j K(O, Y)$. To do this, we note that

$$K(X, Y) = (s_n)^{-1} r^{1-n} \sum_{k=0}^{\infty} N(n, k)(|X|/r)^k P_{k,n}(\xi) \quad (X \in B(r)\backslash\{O\},\ Y \in S(r)) \tag{6}$$

where $N(n, k)$ is the number of linearly independent homogeneous harmonic polynomials of degree k in $\mathbf{R}^n$, ξ is the cosine of the angle between X and Y (i.e.,

$$\xi = (r\,|X|)^{-1}(x_1 y_1 + \cdots + x_n y_n),$$

and $P_{k,n}$ is the generalized Legendre polynomial of degree k (see Müller [8], especially Lemma 17, from which (6) can be derived.) Let

$$I(r) = \{X \in E^1 \cap B(r);\ x_1 > 0\} \qquad (r > 0).$$

Observe that if Y is a fixed point of $S(r)$, then ξ remains constant as X varies on $I(r)$. Hence, assuming for the moment that repeated term-by-term differentiation of the series in (6) is valid, we have

$$D^j K(X, Y) = (s_n)^{-1} r^{1-n} \sum_{k=j}^{\infty} N(n, k) k!\, [(k-j)!]^{-1}\, |X|^{k-j}\, r^{-k} P_{k,n}(\xi)$$
$$(j = 1, 2, \ldots, X \in I(r),\ Y \in S(r)). \tag{7}$$

Now since

$$|P_{k,n}(t)| \leqslant 1 \quad (-1 \leqslant t \leqslant 1) \qquad [8, \text{p. } 15] \tag{8}$$

and

$$N(n, k) = O(k^{n-2}) \quad (k \to \infty) \qquad [8, \text{p. } 3] \tag{9}$$

the series in (7) is uniformly convergent in $I(\rho) \times S(r)$ for each ρ such that $0 < \rho < r$, and hence the term-by-term differentiation is justified. Taking the limit in (7) as X tends to O along $I(r)$ and using the continuity of D^jK, we obtain in view of (8) and (9)

$$\begin{aligned} |D^jK(O, Y)| &\leqslant (s_n)^{-1} r^{1-n-j} N(n, j)\, j! \qquad (j = 1, 2, \ldots, Y \in S(r)) \\ &\leqslant A(s_n)^{-1} j!\, j^{n-2} r^{1-n-j}, \end{aligned} \tag{10}$$

where A depends only on n.

Returning to (5), we now find, using (10) and the hypothesis (4), that

$$|D^jh(O)| \leqslant ACj!\, j^{n-2} r^{-j} e^{\alpha r}, \tag{11}$$

where C is the constant implied by the O-notation in (4). In particular, taking $r = j/\alpha$ in (11), we have

$$\begin{aligned} |D^jh(O)| &= O(j!\, j^{n-2-j}(\alpha e)^j) \qquad (j \to \infty) \\ &= O(j^{n-3/2}\alpha^j), \end{aligned}$$

by Stirling's formula.

3. Proof of Theorem 1

Theorem A enters the proof of Theorem 1 via the following result.

Lemma 1. *Let (a_j) be a real sequence such that*

$$a_j = O(\beta^j/j!) \qquad (j \to \infty), \tag{12}$$

where $0 < \beta < \pi$, and let

$$g(t) = \sum_{j=0}^{\infty} a_j t^j \qquad (t \in \mathbf{R}). \tag{13}$$

If $g(t) = 0$ for all non-negative integers t, then $g \equiv 0$.

To prove the lemma, define f in $\mathbf{C}$ by putting

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

so that $f(z) = g(z)$, when z is real. From (12) it follows that

$$f(z) = O(e^{\beta|z|})$$

uniformly as $|z| \to \infty$, so that, since $f(z) = g(z) = 0$ when z is a non-negative integer, we have, by Theorem A, $f \equiv 0$ and hence $g \equiv 0$.

Next we note the following simple consequence of the maximum principle.

LEMMA 2. *If h is a harmonic function of exponential type α in $\mathbf{R}^n$, then the value of α is unaffected by a translation of the axes.*

The proof of the first paragraph of the theorem is by induction on m, and we deal first with the case $m = 1$. Since h is given in $\mathbf{R}^n$ by its multiple Taylor series about O we have in partcular.

$$h(x_1, 0, \ldots, 0) = \sum_{j=0}^{\infty} D^j h(O)(j!)^{-1} x_1{}^j \qquad (x_1 \in \mathbf{R}). \tag{14}$$

Since h is of exponential type less than π, there is a number α with $0 < \alpha < \pi$ such that

$$M(h; r) = O(e^{\alpha r}) \qquad (r \to \infty).$$

By Theorem 6,

$$D^j h(O) = o(\beta^j) \qquad (j \to \infty), \tag{15}$$

where β is any number such that $\alpha < \beta < \pi$. Applying Lemma 1 with $a_j = D^j h(O)/j!$, we see that $h = 0$ on E^1.

Now suppose that $1 \leqslant m < n$, that the theorem is true for m and that h satisfies the hypotheses of the theorem for $m + 1$. Using the theorem for m, we find that $h = 0$ on E^m. Similarly, using Lemma 2, we obtain by translating the origin along the x_{m+1}-axis that $h = 0$ on each of the sets

$$\{X \in E^{m+1}: x_{m+1} = k\} \qquad (k = 0, 1, 2, \ldots)$$

and hence on their union F^m, say. Hence if J is a doubly infinite line in E^{m+1} parallel to the x_{m+1}-axis, $h(X) = 0$ whenever $X \in J$ and $x_{m+1} \in \mathbf{N}$. By translating the axes so that J coincides with the x_{m+1}-axis, we obtain by a trivial generalization of the case $m = 1$ (interchange the roles of x_1 and x_{m+1}) of the theorem that $h = 0$ on J. Since this holds for each such J, $h = 0$ on E^{m+1}, and the induction is complete.

The second paragraph of the theorem may be proved by considering the function u, defined in $\mathbf{R}^n$ by

$$u(X) = \sin \pi x_1 \cosh \pi x_n\,. \tag{16}$$

It is clear that u is a harmonic function of exponential type π in $\mathbf{R}^n$ and that $u(X) = 0$ if and only if $x_1 \in \mathbf{Z}$. Hence $h = 0$ on L^m for each m, but h does not vanish identically on any E^m.

4. Proofs of Theorems 2, 3 and 4

If h satisfies the hypotheses of Theorem 2, then by the case $m = n - 1$ of Theorem 1, $h = 0$ on E^{n-1}. By a translation argument of the type used in the proof of Theorem 1, we also have $h = 0$ on the set $\{X \in \mathbf{R}^n : x_n = 1\}$. Repeated use of the reflection principle and the uniqueness of harmonic continuations gives $h = 0$ on $F^{n-1} = \{X \in \mathbf{R}^n : x_n \in \mathbf{N}\}$. In particular, $h = 0$ on L^n, so that by the case $m = n$ of Theorem 1, $h \equiv 0$.

If h satisfies the hypotheses of Theorem 3, then by a trivial generalization of the case $m = n - 1$ of Theorem 1, $h = 0$ on each of the hyperplanes

$$\{X \in \mathbf{R}^n : x_k = 0\} \qquad (k = 1, \dots, n).$$

By n applications of the reflection principle and the uniqueness of harmonic continuations, we find that

$$h(X) = (-1)^n h(-X) \qquad (X \in \mathbf{R}^n),$$

which together with the hypothesis (3) implies that $h \equiv 0$.

To prove Theorem 4, we need the following simple result.

Lemma 3. *If h is a harmonic function of exponential type α in $\mathbf{R}^n$, then $\partial h/\partial x_n$ is of exponential type at most α.*

To prove the lemma suppose that $r > 0$ and that $Y \in S(r)$. Then

$$\left| \frac{\partial h}{\partial x_n}(Y) \right| \leqslant c \sup_{|X-Y|=1} |h(X)| \leqslant c\mathcal{M}(h; r+1),$$

where c depends only on n. For the first inequality see e.g. [3, p. 198]; the second follows from the maximum principle. These inequalities imply that

$$\mathcal{M}(\partial h/\partial x_n ; r) \leqslant c\mathcal{M}(h; r+1),$$

whence the lemma follows.

Suppose now that h satisfies the hypotheses of Theorem 4. By the case $m = n - 1$ of Theorem 1, $h = 0$ on E^{n-1}. Similarly, by Lemmas 2 and 3 and the case $m = n - 1$ of Theorem 1, $\partial h/\partial x_n = 0$ on E^{n-1}. A simple argument involving the reflection principle shows that all the partial derivatives of h are zero on E^{n-1} and hence that $h \equiv 0$.

5. Proof of Theorem 5

We argue as in the proof of Theorem 1, using in place of Lemma 1 the following consequence of Theorem C.

LEMMA 4. *Let (a_j) be a real sequence satisfying* (12) *for some β such that $0 < \beta < \pi$, and let g be given by* (13). *If there is a number M such that $|g(t)| \leqslant M$ whenever t is an integer, then $|g(t)| \leqslant \lambda(\beta)M$ for all real t.*

This follows from Theorem C in the same way that Lemma 1 follows from Theorem A.

Suppose that h satisfies the hypotheses of Theorem 5 in the case $m = 1$. If β is a number such that $\alpha < \beta < \pi$, then

$$\mathscr{M}(h; r) = o(e^{(\alpha+\beta)r/2}) \qquad (r \to \infty),$$

and hence, by Theorem 6, (15) holds. Using (15), (16) and Lemma 4, we find that $|h| \leqslant \lambda(\beta)M$ on E^1. Letting $\beta \to \alpha+$ and using the continuity of λ, we have $|h| \leqslant \lambda(\alpha)M$ on E^1.

Next suppose that $1 \leqslant m < n$, that the theorem is true for m and that h satisfies the hypotheses of the theorem for $m + 1$. Using the theorem for m and proceeding as in the proof of Theorem 1, we find that $|h| \leqslant \{\lambda(\alpha)\}^m M$ on the set $\{X \in E^{m+1}: x_{m+1} \in \mathbf{Z}\}$. In particular, if J is a doubly infinite line parallel to the x_{m+1}-axis, then $|h(X)| \leqslant \{\lambda(\alpha)\}^m M$ whenever $X \in J$ and $x_{m+1} \in \mathbf{Z}$. Hence, by a trivial generalization of the case $m = 1$ of the theorem, we have $|h(X)| \leqslant \{\lambda(\alpha)\}^{m+1} M$ whenever $X \in J$. Since this inequality holds on all such lines J, it holds on E^{m+1}. By induction, the proof of the first paragraph of Theorem 5 is complete.

Consideration of the function u, defined in $\mathbf{R}^n$ by (16) proves the second paragraph of the theorem in the case $m = n$. For the other cases, consider the function v, defined in $\mathbf{R}^n$ by

$$v(X) = e^{\pi x_n}(x_1 \sin \pi x_1 - x_n \cos \pi x_n).$$

It is easy to show that v is a harmonic function in $\mathbf{R}^n$ of exponential type π. Further, $v = 0$ on the set $\{X \in \mathbf{R}^n: x_1 \in \mathbf{Z}, x_n = 0\}$ which contains L^m $(m = 1, 2,..., n - 1)$. However, v is unbounded on E^1 and hence on every E^m.

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